L'Hospital-Bernoullijevo pravilo

Ako su obe f-je $f(x)$ i $g(x)$ beskonačno male ili beskonačno velike kad $x \to a$ tj. ako razlomak $\frac{f(x)}{g(x)}$ predstavlja u tački $x=a$ neodređen oblik tipa $\frac{0}{0}$ ili $\frac{\infty}{\infty}$ tada je
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

Neodređene limese koji su oblika $0\cdot\infty$, $\infty-\infty$, 1^∞, 0^0, ∞^0 skoro uvijek možemo svesti na neki od oblika $\frac{0}{0}$ ili $\frac{\infty}{\infty}$ i onda ih naći pomoću Lopitalovog pravila.

Izračunati:

1. $$\lim_{x\to 0}\frac{\ln x}{\operatorname{ctg} x}\left(\frac{-\infty}{\infty}\right) \overset{LoP.}{=} \lim_{x\to 0}\frac{(\ln x)'}{(\operatorname{ctg} x)'} = \lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{\sin^2 x}} = -\lim_{x\to 0}\frac{\sin^2 x}{x} =$$
$$= -\lim_{x\to 0}\frac{\sin x}{x}\cdot\sin x = -1\cdot 0 = 0$$

2. $$\lim_{x\to 1}\frac{x^3-2x^2-x+2}{x^3-7x+6}\left(\frac{0}{0}\right) \overset{LoP.}{=} \lim_{x\to 1}\frac{3x^2-4x-1}{3x^2-7} = \frac{-2}{-4} = \frac{1}{2}$$

3. $$\lim_{x\to 0}\frac{x\cos x-\sin x}{x^3}\left(\frac{0}{0}\right) \overset{LoP.}{=} \lim_{x\to 0}\frac{\overbrace{\cos x + x(-\sin x) - \cos x}^{-x\sin x}}{3x^2}\left(\frac{0}{0}\right) \overset{LoP}{=} \lim_{x\to 0}\frac{-\sin x+(-x)\cos x}{6x}$$
$$\left(\frac{0}{0}\right) \overset{LoP.}{=} \lim_{x\to 0}\frac{-\cos x-\cos x-x(-\sin x)}{6} = \frac{-2}{6} = -\frac{1}{3}$$

4. $$\lim_{x\to 1}\frac{1-x}{1-\sin\frac{\pi x}{2}}\left(\frac{0}{0}\right) \overset{LoP.}{=} \lim_{x\to 1}\frac{-1}{-\cos\frac{\pi x}{2}\cdot\frac{\pi}{2}} = \frac{-1}{-0} = +\infty$$

5. $$\lim_{x\to 0}\frac{\operatorname{tg} x-\sin x}{x-\sin x}\left(\frac{0}{0}\right) \overset{LoP.}{=} \lim_{x\to 0}\frac{\frac{1}{\cos^2 x}-\cos x}{1-\cos x} = \lim_{x\to 0}\frac{\frac{1-\cos^3 x}{\cos^2 x}}{1-\cos x}$$
$$= \lim_{x\to 0}\frac{(1-\cos x)(1+\cos x+\cos^2 x)}{\cos^2 x(1-\cos x)} = 3$$

6. $$\lim_{x\to 0}\frac{\sin 5x}{x}\left(\frac{0}{0}\right) \overset{LoP.}{=} \lim_{x\to 0}\frac{\cos 5x\cdot 5}{1} = 5$$

7. $\lim\limits_{x\to\infty} \frac{e^x}{x^5} \left(\frac{\infty}{\infty}\right) \overset{L.P.}{=} \lim\limits_{x\to\infty} \frac{e^x}{5x^4} \left(\frac{\infty}{\infty}\right) \overset{L.P.}{=} \lim\limits_{x\to\infty} \frac{e^x}{20x^3} \left(\frac{\infty}{\infty}\right) \overset{L.P.}{=} \dots = \frac{\infty}{120} = \infty$

8. $\lim\limits_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} \left(\frac{\infty}{\infty}\right) \overset{L.P.}{=} \lim\limits_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{3\sqrt[3]{x^2}}} = \lim\limits_{x\to\infty} \frac{3\sqrt[3]{x^2}}{x} = 3\lim\limits_{x\to\infty} \frac{x^{\frac{2}{3}} : x}{x \; : x} = 3\lim\limits_{x\to\infty} x^{-\frac{1}{3}} = 0$

9. $^{\vee}$ $\lim\limits_{x\to 0} \frac{\ln(\sin mx)}{\ln \sin x}$ Rj. 1

10. $\lim\limits_{x\to 1} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right) \; (\infty - \infty) = \lim\limits_{x\to 1} \frac{\ln x - (x-1)}{(x-1)\ln x} \left(\frac{0}{0}\right) \overset{L.P.}{=}$

$= \lim\limits_{x\to 1} \frac{\frac{1}{x} - 1}{\ln x + (x-1)\frac{1}{x}} = \lim\limits_{x\to 1} \frac{\frac{1}{x} - 1}{\ln x - \frac{1}{x} + 1} \left(\frac{0}{0}\right) \overset{L.P.}{=} \lim\limits_{x\to 1} \frac{-\frac{1}{x^2}}{\frac{1}{x} + \frac{1}{x^2}} = -\frac{1}{2}$

11. $\lim\limits_{x\to 0} (1-\cos x)\operatorname{ctg} x \; (0\cdot\infty) = \lim\limits_{x\to 0} \frac{(1-\cos x)\cos x}{\sin x} \left(\frac{0}{0}\right) =$

$= \lim\limits_{x\to 0} \frac{1-\cos x}{\sin x} \cdot \lim\limits_{x\to 0} \cos x \overset{L.P.}{=} \lim\limits_{x\to 0} \frac{\sin x}{\cos x} \cdot \lim\limits_{x\to 0} \cos x = 0 \cdot 1 = 0$

12. $\lim\limits_{x\to\infty} \left[x \cdot \left(e^{-\frac{2}{x}} - 1\right)\right] \; (\infty \cdot 0) = \lim\limits_{x\to\infty} \frac{e^{-\frac{2}{x}} - 1}{\frac{1}{x}} \left(\frac{0}{0}\right) \overset{L.P.}{=} \lim\limits_{x\to\infty} \frac{e^{-\frac{2}{x}} \cdot (-2)\cdot(-1)\cdot x^{-2}}{(-1)\cdot x^{-2}}$

$= e^0 \cdot (-2) = -2$

13. $^{\vee}$ $\lim\limits_{x\to\infty} x \cdot \sin\frac{a}{x}$ Rj. a

14. $\lim\limits_{x\to 1} x^{\frac{1}{1-x}} \; (1^\infty) = \lim\limits_{x\to 1} e^{\ln x^{\frac{1}{1-x}}} = \lim\limits_{x\to 1} e^{\frac{1}{1-x}\cdot \ln x} = e^{\lim\limits_{x\to 1} \frac{\ln x}{1-x}} \left(\frac{0}{0}\right) =$

$\overset{L.P.}{=} e^{\lim\limits_{x\to 1} \frac{\frac{1}{x}}{-1}} = e^{-1} = \frac{1}{e}$

15. $\lim\limits_{x\to 0} (\operatorname{ctg} x)^{\frac{1}{\ln x}} \; (\infty^\infty) = \lim\limits_{x\to 0} e^{\ln(\operatorname{ctg} x)^{\frac{1}{\ln x}}} = e^{\lim\limits_{x\to 0} \frac{\ln(\operatorname{ctg} x)}{\ln x}} \left(\frac{\infty}{\infty}\right)$

$\overset{L.P.}{=} e^{\lim\limits_{x\to 0} \frac{\frac{1}{\operatorname{ctg} x} \cdot \frac{-1}{\sin^2 x}}{\frac{1}{x}}} = e^{\lim\limits_{x\to 0} \frac{-\frac{\sin x}{\cos x} \cdot \frac{1}{\sin^2 x}}{\frac{1}{x}}} = e^{\lim\limits_{x\to 0} \frac{-x}{\sin x \cos x}} \left(\frac{0}{0}\right) \overset{L.P.}{=} e^{\lim\limits_{x\to 0} \frac{-1}{\cos^2 x - \sin^2 x}}$

$= e^{-1} = \frac{1}{e}$

16. $^{\vee}$ $\lim\limits_{x\to 0} x^{\sin x}$ Rj. 1

17. $^{\vee}$ $\lim\limits_{x\to\infty} \left[(x-1)e^{\frac{-1}{x+1}} - x\right]$ Rj. -2

Ako je $h(x)=\frac{1}{\sin x}-\frac{1}{x}$ izračunati $\lim\limits_{x\to 0} h'(x)$.

Rj. $h(x)=\frac{1}{\sin x}-\frac{1}{x}=\frac{x-\sin x}{x\sin x}$

$$h'(x)=\left(\frac{1}{\sin x}\right)'-\left(\frac{1}{x}\right)'=(\sin^{-1}x)'-(x^{-1})'=(-1)\sin^{-2}x\cdot\cos x-(-1)x^{-2}$$

$$h'(x)=\frac{-\cos x}{\sin^2 x}+\frac{1}{x^2}=\frac{1}{x^2}-\frac{\cos x}{\sin^2 x}=\frac{\sin^2 x-x^2\cos x}{x^2\sin^2 x}$$

$$\lim_{x\to 0} h'(x)=\lim_{x\to 0}\frac{\sin^2 x-x^2\cos x}{x^2\sin^2 x}\left(=\frac{0}{0}\right)\overset{L.P.}{=}$$

$$\overset{L.P.}{=}\lim_{x\to 0}\frac{\overbrace{2\sin x\cos x}^{\sin 2x}-(2x\cos x+x^2(-\sin x))}{2x\sin^2 x+x^2\underbrace{2\sin x\cos x}_{\sin 2x}}=\lim_{x\to 0}\frac{\sin 2x-2x\cos x+x^2\sin x}{2x\sin^2 x+x^2\sin 2x}$$

$$\left(=\frac{0}{0}\right)\overset{L.P.}{=}\lim_{x\to 0}\frac{\cos 2x\cdot 2-2(\cos x+x(-\sin x))+(2x\sin x+x^2\cos x)}{2(\sin^2 x+x\underbrace{2\sin x\cos x}_{\sin 2x})+2x\sin 2x+x^2\cos 2x\cdot 2}=$$

$$=\lim_{x\to 0}\frac{2\cos 2x-2\cos x+2x\sin x+2x\sin x+x^2\cos x}{2\sin^2 x+2x\sin 2x+2x\sin 2x+2x^2\cos 2x}=$$

$$=\lim_{x\to 0}\frac{2\cos 2x-2\cos x+x^2\cos x+4x\sin x}{2\sin^2 x+2x^2\cos 2x+4x\sin 2x}$$

$$=\lim_{x\to 0}\frac{2(-\sin 2x)\cdot 2-2(-\sin x)+(2x\cos x+x^2(-\sin x))+4\sin x+4x\cos x}{2\cdot\underbrace{2\sin x\cos x}_{\sin 2x}+2(2x\cos 2x+x^2(-\sin 2x)\cdot 2)+4\sin 2x+4x\cos 2x\cdot 2}$$

$$=\lim_{x\to 0}\frac{-4\sin 2x+6\sin x+6x\cos x-x^2\sin x}{6\sin 2x+12x\cos 2x-4x^2\sin 2x}\left(=\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{-4\cos 2x\cdot 2+6\cos x+6(\cos x+x(-\sin x))\cdot(2x\sin x+x^2\cos x)}{6\cos 2x\cdot 2+12(\cos 2x+x(-\sin 2x)\cdot 2)-4(2x\sin 2x+x^2\cos 2x\cdot 2)}=$$

$$=\frac{-8+6+6}{12+12}=\frac{4}{24}=\frac{1}{6}$$

Prema tome $\lim\limits_{x\to 0} h'(x)=\frac{1}{6}$

(Zadaci su skinuti sa stranice: \pf.unze.ba\nabokov
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